Determinant Formulas

Let's use the axioms to derive $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

Axiom3

$$= a\det\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} + b\det\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

Axiom3

Axiom3 →

$$= a\left(c\det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + b\left(c\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

$\det\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 0$ by Prop 2

$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ by Axiom 1

$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$ by Axiom 2

$\det\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0$ by Prop 2

$$= ad - bc$$

We can do this to derive a formula the determinant for any square matrix

Cofactor expansions repackage this into a recursive formula

Def: Let A be an $n \times n$ matrix,

Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by removing the ith row and jth column of A

Let $C_{ij} = (-1)^{i+j} \det A_{ij}$. this is a cofactor of A

Recursive Formula: $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

cofactor expansion along the first row

Ex: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $C_{11} = (-1)^{1+1}\det[d] = d$

$C_{12} = (-1)^{1+2}\det[c] = -c$

So $\det A = aC_{11} + bC_{12} = ad - bc$ ✓

Note: 1×1 determinants: $\det[x] = |x|$

Ex: $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + (-1)(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 = 1 + 1 = 2$

↑ vertical bars mean determinant.

Ex:

$$\begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 5 & 2 & -1 & 8 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 8 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 5 & -1 & 8 \end{vmatrix} + (-1)(-1)^{1+3}\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 5 & 2 & 8 \end{vmatrix}$$

Now do 3×3 determinants!

Well this is tedious.

To make your life easier, you can perform a cofactor expansion along any row

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$$ (cofactor expansion along i^{th} row)

Ex: Redo last example, but faster

$$\begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \\ 5 & 2 & -1 & 8 \end{vmatrix} = 3(-1)^{2+3}\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 5 & 2 & 8 \end{vmatrix} = -3\left(1\begin{vmatrix} 1 & 1 \\ 2 & 8 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 5 & 8 \end{vmatrix}\right)$$

$$= -3(6-6) = 0$$

Many zeros

You can also expand along any column!

$$\det A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$ (cofactor expansion along j^{th} column)

You can combine these in a single calculation

Ex:
$$\begin{vmatrix} 1 & 2 & 3 & 0 & 5 \\ 6 & 7 & 8 & 9 & 9 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 5 \\ 0 & 9 & 2 & 0 & 6 \end{vmatrix} = -1\begin{vmatrix} 1 & 3 & 0 & 5 \\ 6 & 8 & 9 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & 0 & 6 \end{vmatrix} = (-1)(-9)\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 5 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= (-1)(-9)(1)\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix}$$

$$= (-1)(-9)(1)(-4)$$

$$= -36$$

Using cofactors to find inverses

Recall the 2×2 inverse formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\det}} \begin{bmatrix} \overset{C_{11}}{\downarrow} d & \overset{C_{21}}{\downarrow} -b \\ -c & a \\ \underset{C_{12}}{\uparrow} & \underset{C_{22}}{\uparrow} \end{bmatrix}$$

In general, if A is $n \times n$ and invertible, then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & \ddots & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det A} C^T$$

(C is called the cofactor matrix, C^T is called the adjugate)

Proof idea: $A^{-1} = \frac{1}{\det A} C^T \iff I = \frac{1}{\det A} AC^T$

$$\iff AC^T = (\det A) I$$

So show $AC^T = (\det A) I$ (recitation) ∎

Using determinants to solve systems

We want to solve $Ax=b$.

Let $A=[a_1 \cdots a_n]$.

Let $B_j = [a_1 \cdots a_{j-1}\ b\ a_{j+1} \cdots a_n]$ (replace j^{th} row of A by b)

Cramer's Rule: If $\det A \neq 0$, then $Ax=b$ has solution $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where $x_i = \dfrac{\det B_i}{\det A}$

Proof:

Let $I_j = [e_1 \cdots e_{j-1}\ b\ e_{j+1} \cdots e_n]$

Then for $j=1$,

$$AI_1 = [a_1 \cdots a_n] \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & 1 & & 0 \\ x_n & 0 & \ddots & 1 \end{bmatrix} = \left[A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ Ae_2 \cdots Ae_n \right]$$

$$= [b\ a_2 \cdots a_n] = B_1$$

In general, $AI_j = B_j$

Notice $\det I_j = x_j$ (use cofactor expansion along j^{th} row)

Hence $\det B_j = \det(AI_j) = (\det A)(\det I_j) = x_j \det A$

$$\Rightarrow \quad x_j = \frac{\det B_j}{\det A}$$

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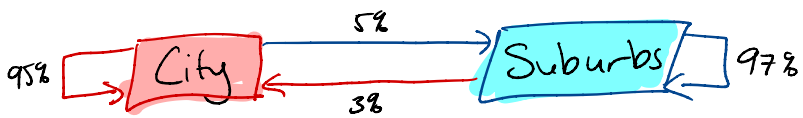
Markov Chain Example

Let's consider the population migration between a city and its suburbs. Suppose that every year:

- 95% of the city population stays in the city
- 5% of the city population moves to the suburbs
- 97% of the suburb population stays in the suburbs
- 3% of the suburb population moves to the city



Suppose the population of the city is currently 600,000 and the population of the suburbs is 400,000.

a) What will the populations be next year?

City: $.95(600{,}000) + .03(400{,}000) = 582{,}000$

Suburbs: $.05(600{,}000) + .97(400{,}000) = 418{,}000$

We can encode this using matrix-vector multiplication. First the migration data can be put in a matrix:

$$M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$$

To: City, Subs

From: City, Subs

Called a Markov Matrix

(Square matrix w/ nonneg. entries such that each column has entries adding to 1)

The initial populations can be put into a vector

$$x_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \begin{matrix} \leftarrow \text{percentage of pop. in city} \\ \leftarrow \text{percentage of pop. in suburbs} \end{matrix}$$

called the initial state vector.

Then $x_1 = Mx_0 = \begin{bmatrix} .582 \\ .418 \end{bmatrix}$ contains the percentages of the population in one year.

b) What are the populations after k years?

$$x_k = M^k x_0 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}^k \begin{bmatrix} .6 \\ .4 \end{bmatrix} = ??$$

$x_k = M_k x_0$ is called a Markov Chain

c) What happens in the long run (as $k \to \infty$)?

To answer b) and c), we must develop the theory of eigenvalues and eigenvectors.